One step Binomial tree

Let us consider the example with the tree diagram illustrated in the previous slide. Let the portfolio consist of long position in Δ shares of stock and a short position in one call option. We will find the value of Δ which would make the portfolio riskless.

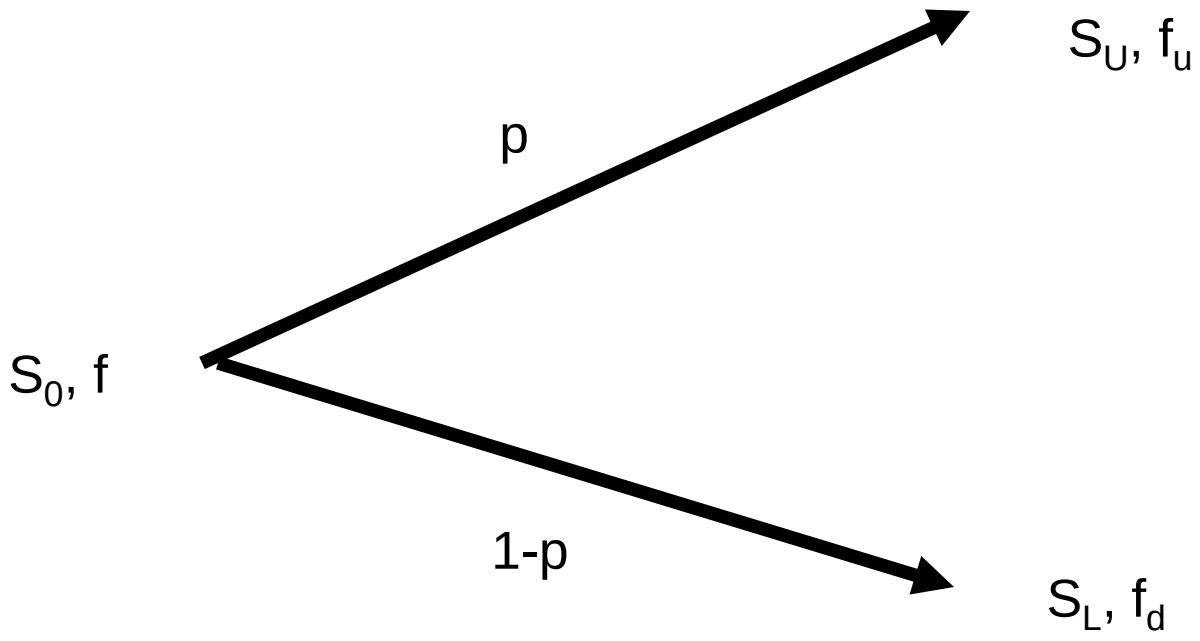
For upward price movement the value of portfolio is $22*\Delta-1$

For downward price movement the value of portfolio is $18*\Delta$

Thus $22*\Delta-1= 18*\Delta$, i.e., $\Delta=0.25$

One step Binomial tree

Consider the stock prices can either move up or down, one step



One step Binomial tree

Let the portfolio consist of long position in Δ shares (price S_0) of stock and a short position in one option (price f). We find the value of Δ which would make the portfolio riskless.

- For upward price movement (u>1), (i) the price of stock is S_0*u and (ii) payoff from option is f_u. Hence the value of portfolio is $S_0*u*\Delta-f_u$
- For downward price movement (d<1), (i) the price of stock is S_0*d and (ii) payoff from option is f_d. Hence the value of portfolio is $S_0*d*\Delta-f_d$

One step Binomial tree

- For no arbitrage we have $S_0*u*\Delta - f_u = S_0*d*\Delta - f_d$, hence $\Delta = (f_u - f_d)/(S_0*u - S_0*d)$
- Also if risk free interest rate is r, then we must have $S_0*\Delta - f = (S_0*u*\Delta - f_u)e^{-rT}$
- Hence $f = e^{-rT}[p*f_u + (1-p)*f_d]$, where $p=(e^{rT}-d)/(u-d)$

Risk Neutral valuation

$$E(S_T) = S_0 * e^{rT}$$

Example 9.1

A stock price is currently $40. It is known that at the end of one month it will be either $42 or $38. The risk free interest rate is 8% per annum with continuous compounding. What is the value of a one month European call option with a strike price of $39?

Example 9.2

Explain the no-arbitrage and risk neutral valuation approaches to valuing a European option using one step binomial tree.

Chapter 10 & 11: Model of the Behaviour of Stock Prices and Black-Scholes model

We will cover and discuss about

- Markov property
- Continuous time stochastic processes
- Ito's Lemma

We will cover and discuss about

- Lognormal property of stock prices
- Distribution of rate of return
- Volatility
- Concepts and derivations of Black-Scholes-Merton differential equation
- Implied volatilities

Markov property

Is a particular type of stochastic process where only the present value of a variable is relevant for predicting the future.

Categorization of Stochastic Processes

- Discrete time; discrete variable
- Discrete time; continuous variable
- Continuous time; discrete variable
- Continuous time; continuous variable

Modeling Stock Prices

We can use any of the four types of stochastic processes to model stock prices, but the continuous time, continuous variable process proves to be the most useful for the purposes of valuing derivatives.

Markov process

In a Markov process future movements in a variable depend only on where we are, not the history of how we got where we are. We assume that stock prices follow Markov processes

Weak-Form Market Efficiency

A Markov process for stock prices is clearly consistent with weak-form market efficiency, which asserts that it is impossible to produce consistently superior returns with a trading rule based on the past history of stock prices. In other words technical analysis does not work.

Discrete Time Continuous Variable Model

A stock price is currently at $350. At the end of 1 year it is considered that it will have a probability distribution of N(350,400) where $N(\mu,\sigma^2)$ is a normal distribution with mean μ and variance σ^2. Then we would wish to answer what is the probability distribution of the stock price at the end of 2 years or ½ years or ¼ years or δt years?

Variances and Standard Deviations

In Markov processes changes in successive periods of time are independent, which means variances are additive but standard deviations are not additive. Thus in our above example it is correct to say that the variance is 800 for two years and not, standard deviation is 40 for the two years.

A Wiener Process

z follows a Wiener process if the following properties hold:

1) The change Δz during a small interval of time Δt is $\Delta z = \varepsilon (\Delta t)^{1/2}$, where $\varepsilon \sim N(0,1)$
2) The values of Δz for any <u>two</u> different (non-overlapping) periods of time are <u>independent</u>.

Hence it follows from the these above two properties that

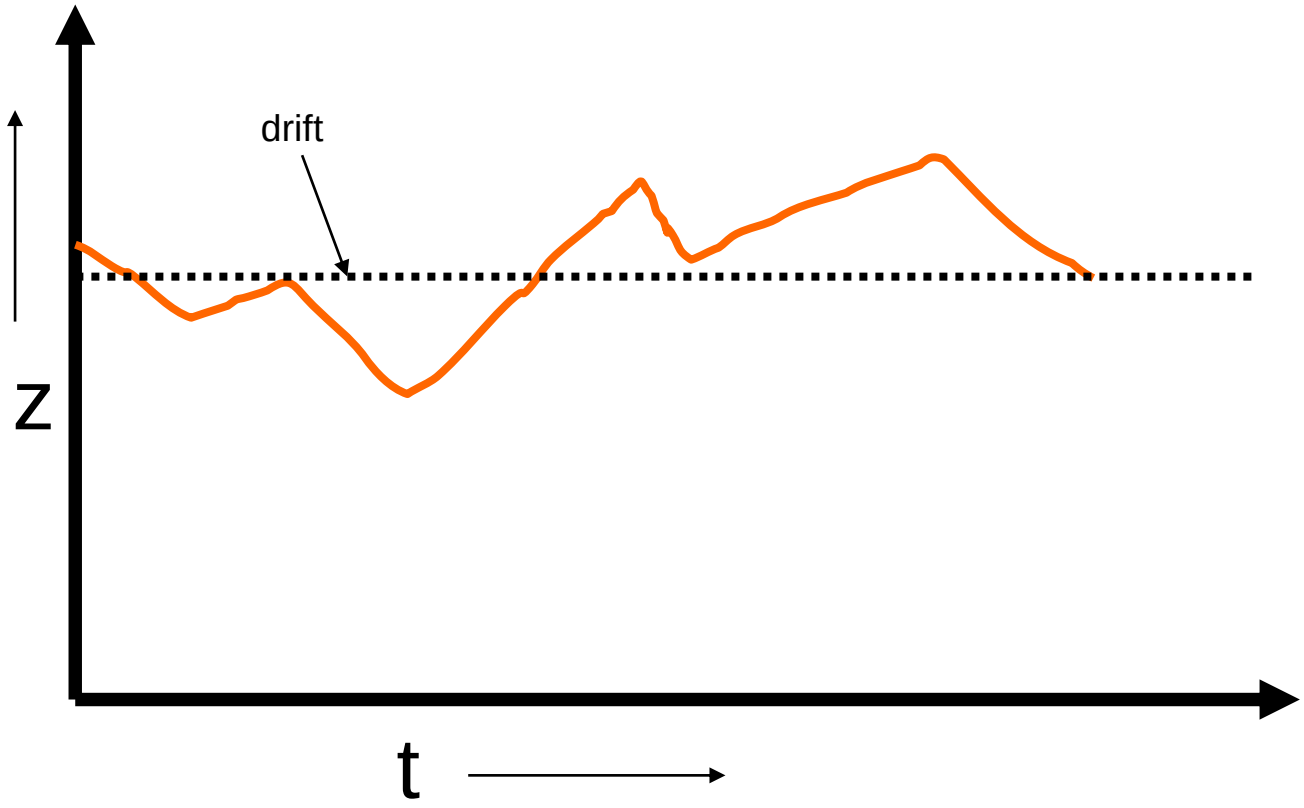
1) $E[\Delta z] = 0$
2) $V[\Delta z] = \Delta t$
3) z follows a Markov process

Properties of a Wiener Process

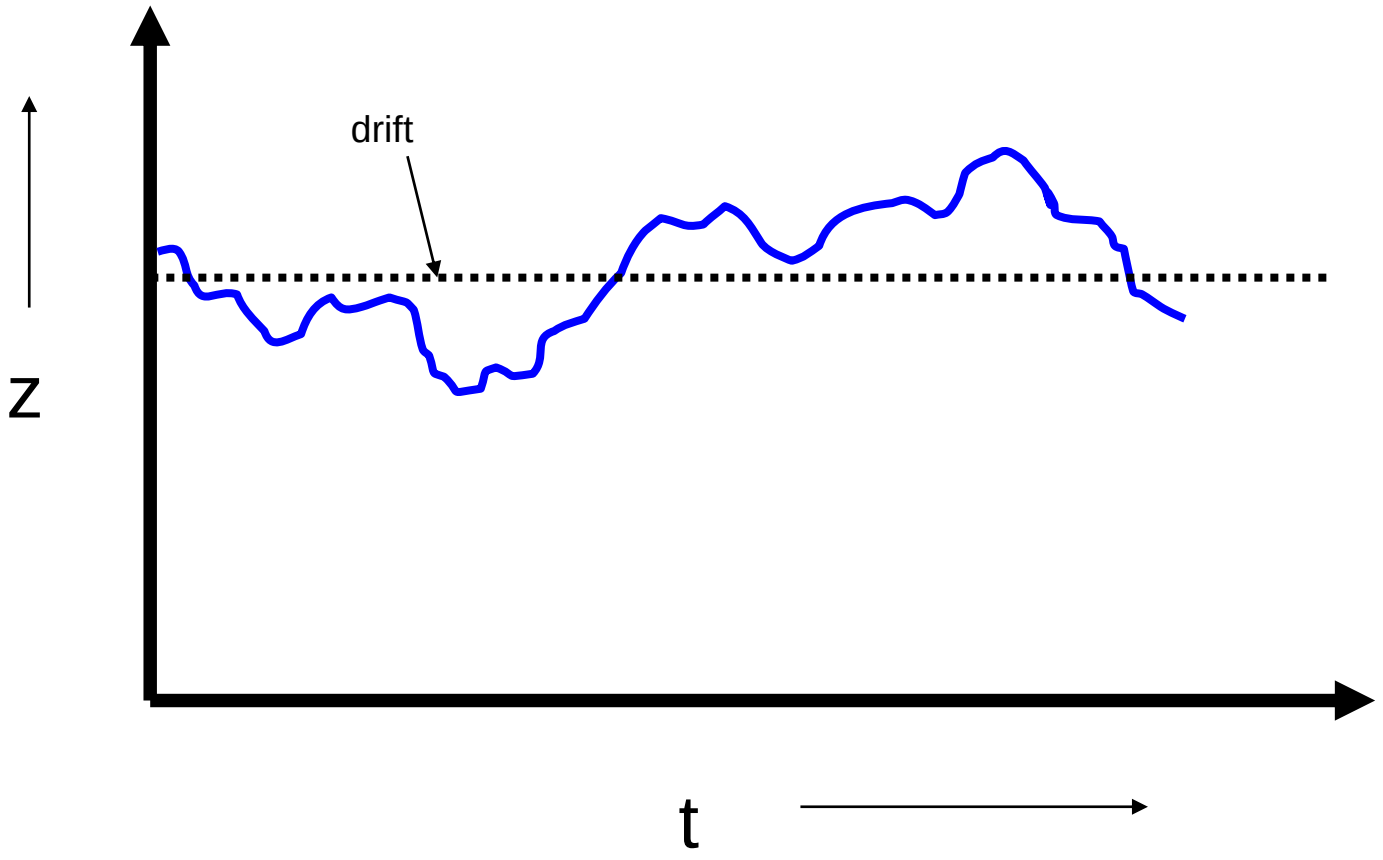
If $T = N\Delta t$, then:

- $E[z(T) - z(0)] = 0$
- $V[z(T) - z(0)] = T$

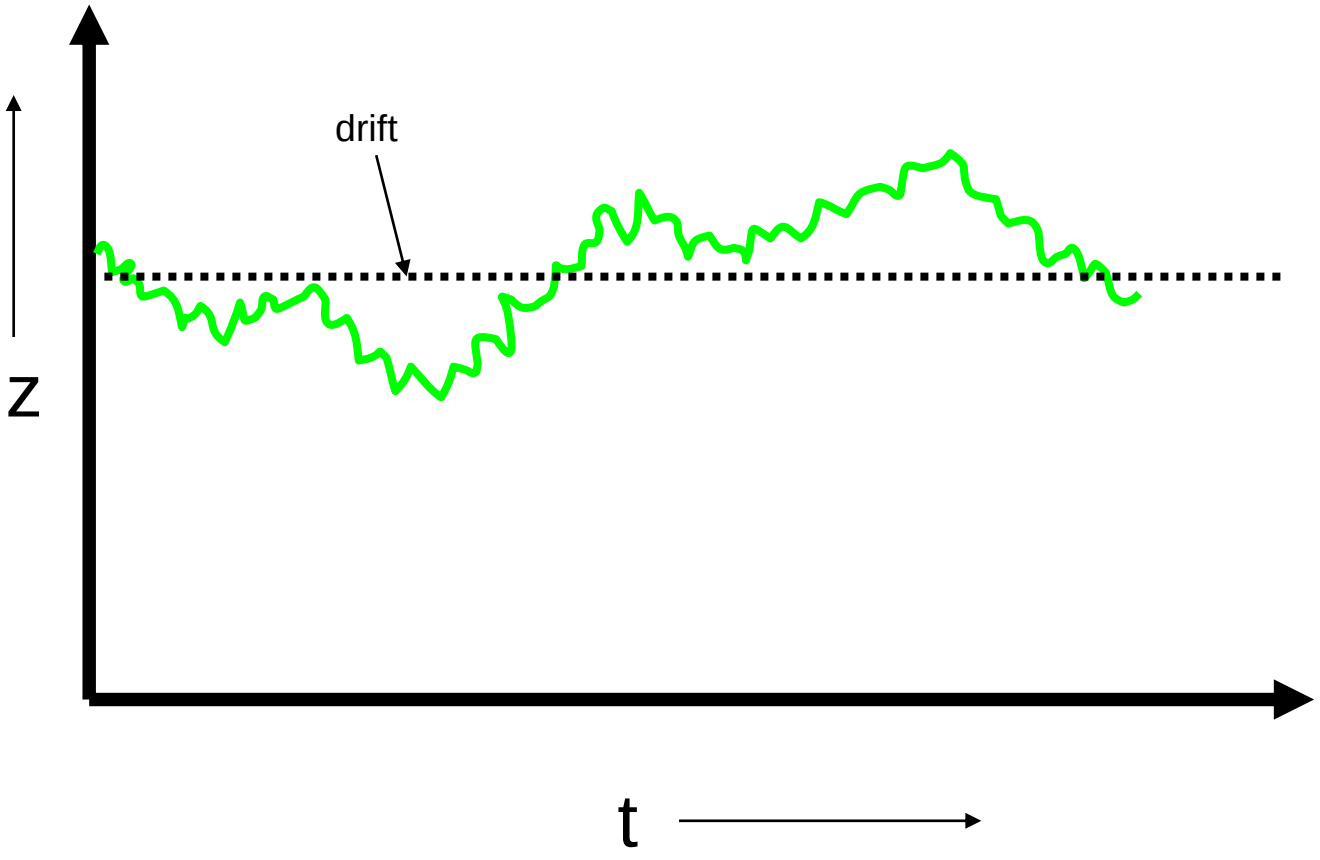
Wiener process



Wiener process



Wiener process



Generalized Wiener Processes

A Wiener process has:

1) Drift rate (i.e. average change per unit time) of 0. The drift rate of 0 means that the expected value of z at any future time is equal to its current value.
2) Variance rate of 1. The variance rate of 1 means that the variance of the change in z in a time interval of length T equals T.

Generalized Wiener Processes

A generalized Wiener process for a variable x can be defined in terms of dz (Wierner process) as dx=a*dt+b*dz, where a an b are constants.

Where:

- a*dt implies that x has an expected drift rate of ′a′ per unit time. If b*dz is not there, then we have $x = x_0 + at$, where x_0 is the value of x at t = 0.
- b*dz implies variability in x. The amount of noise is ′b′ times a Wiener process, which is z.

Generalized Wiener Processes

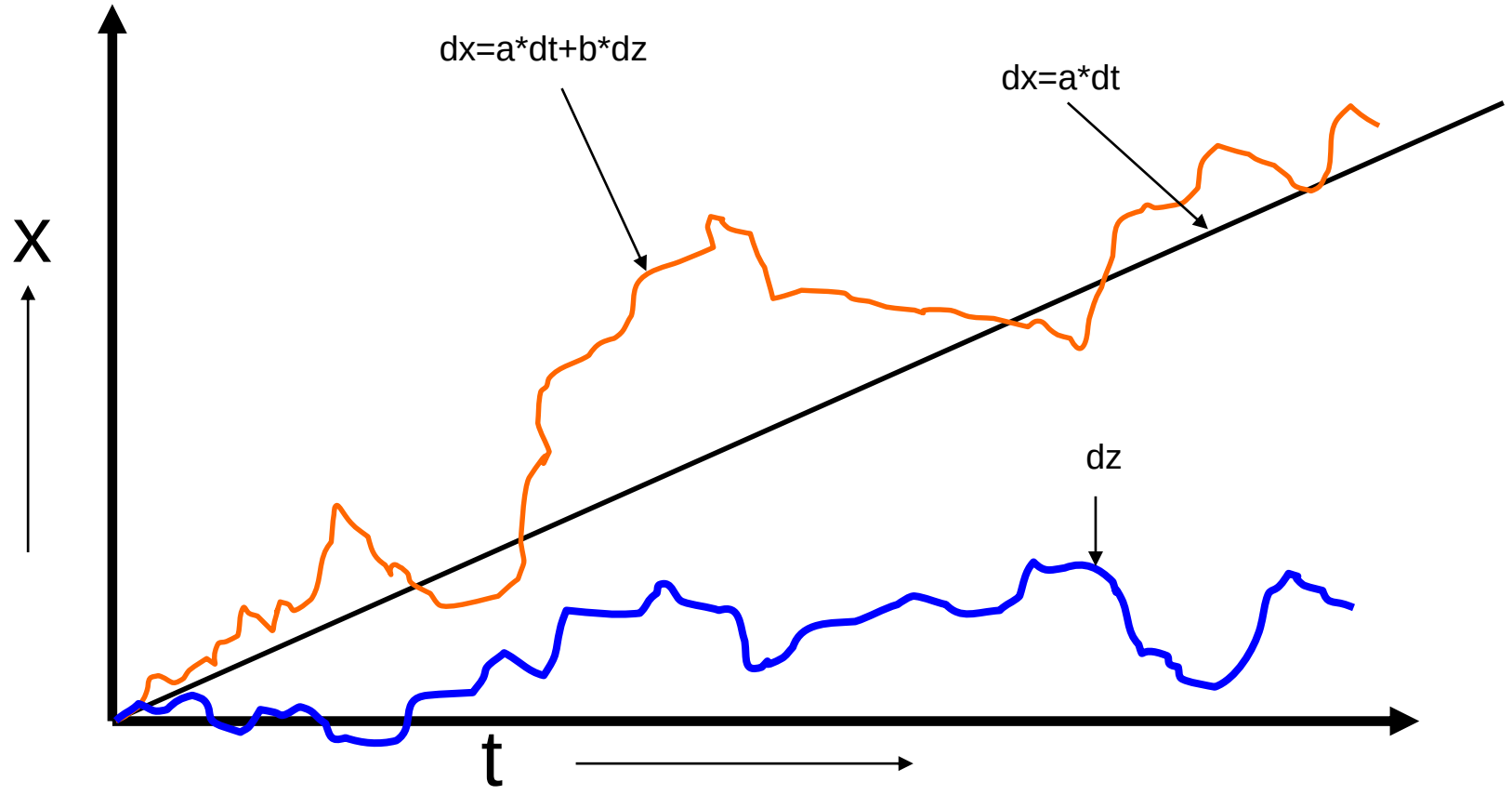
Thus we have:

$\Delta x = a*\Delta t + b*\varepsilon*(\Delta t)^{1/2}$, where $\varepsilon \sim N(0,1)$.

This would imply the following:

- $E[x] = a*\Delta t$
- $V[x] = b^2*\Delta t$
- $E[x(T) - x(0)] = a*T$
- $V[x(T) - x(0)] = b^2*T$

Generalized Wiener Processes



Example

A stock price starts at 40 and has a probability distribution of $\mathrm{N}(40,100)$ at the end of the year

- If we assume the stochastic process is Markov with no drift then the process is dS = 10*dz
- If the stock price were <u>expected to grow by 8 on average during the year</u>, so that the year-end distribution is $\mathrm{N}(48,100)$, the process is dS = 8*dt + 10*dz

Ito Process (a type of generalized Wiener process)

In an Ito process the drift rate and the variance rate are functions of time, i.e., we have:

$$dx = a(x,t)*dt + b(x,t)*dz$$

The discrete time equivalent is

$$\Delta x = a(x,t)*\Delta t + b(x,t)*\varepsilon(\Delta t)^{1/2}$$

Why a Generalized Wiener Process is not Appropriate for Stocks

- For a stock price we can conjecture that its expected percentage change in a short period of time remains constant, not its expected absolute change in a short period of time.
- We can also conjecture that our uncertainty as to the size of future stock price movements is proportional to the level of the stock price.

Ito Process for Stock Prices

$$\Delta S = \mu * S * \Delta t$$

As $\Delta t \rightarrow 0$ and after we integrate we have $S_T = S_0 * \exp(\mu * T)$.

But the question is what about volatility, how can we incorporate it in the model.

If we incorporate volatility, then we have:

$$dS/S = \mu * dt + \sigma * dz$$

This model is sometimes known as Geometric Brownian Motion (GBM).

Ito Process for Stock Prices

The discrete version of the model is:

$$\Delta S/S = \mu*\Delta t + \sigma*\varepsilon*(\Delta t)^{1/2}$$

where:

- $\Delta S = S(t=t_2) - S(t=t_1)$, such that $t_2-t_1=\Delta t$
- μ = expected rate of return per unit time for the stock. [Note: for Δt small we assume μ to be constant].
- σ = volatility of the stock. [Note: for Δt small we assume σ to be constant].
- $\varepsilon \sim N(0,1)$

Stock Prices

$$\Delta S/S \sim N(\mu^*\Delta t , \sigma^{2*}\Delta t)$$

Monte Carlo Simulation

We can sample random paths for the stock price by sampling values for ε

Suppose μ= 0.14, σ= 0.20, and δt = 0.01, then

$$\delta S = 0.0014S + 0.02S\varepsilon$$

Monte Carlo Simulation – One Path

Period	Stock Price at Start of Period	Random Sample for ε	Change in Stock Price, ΔS
0	20.000	0.52	0.236
1	20.236	1.44	0.611
2	20.847	-0.86	-0.329
3	20.518	1.46	0.628
4	21.146	-0.69	-0.262

Ito's Lemma

If we know the stochastic process followed by x, Ito's lemma tells us the stochastic process followed by some function $G(x, t)$

Since a derivative security is a function of the price of the underlying and time, Ito's lemma plays an important part in the analysis of derivative securities

Taylor Series Expansion

A Taylor's series expansion of $G(x, t)$ gives

$$\delta G = \frac{\partial G}{\partial x}\delta x + \frac{\partial G}{\partial t}\delta t + \tfrac{1}{2}\frac{\partial^2 G}{\partial x^2}\delta x^2 + \frac{\partial^2 G}{\partial x \partial t}\delta x\,\delta t + \tfrac{1}{2}\frac{\partial^2 G}{\partial t^2}\delta t^2 + \ldots$$

Ignoring Terms of Higher Order Than δt

In ordinary calculus we have

$$\delta G = \frac{\partial G}{\partial x}\delta x + \frac{\partial G}{\partial t}\delta t$$

In stochastic calculus this becomes

$$\delta G = \frac{\partial G}{\partial x}\delta x + \frac{\partial G}{\partial t}\delta t + \tfrac{1}{2}\frac{\partial^2 G}{\partial x^2}\delta x^2$$

because δx has a component which is of order $\sqrt{\delta t}$

Substituting for δx

Suppose

$$dx = a(x,t)dt + b(x,t)dz$$

so that

$$\delta x = a\,\delta t + b\,\varepsilon\sqrt{\delta t}$$

Then ignoring terms of higher order than δt

$$\delta G = \frac{\partial G}{\partial x}\delta x + \frac{\partial G}{\partial t}\delta t + \tfrac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2\varepsilon^2\delta t$$

The $\varepsilon^2 \Delta t$ *t*erm

Since $\varepsilon \approx \phi(0,1)\ E(\varepsilon) = 0$

$$E(\varepsilon^2) - [E(\varepsilon)]^2 = 1$$

$$E(\varepsilon^2) = 1$$

It follows that $E(\varepsilon^2 \delta t) = \delta t$

The variance of δt is proportional to δt^2 and can be ignored. Hence

$$\delta G = \frac{\partial G}{\partial x}\delta x + \frac{\partial G}{\partial t}\delta t + \frac{1}{2}\frac{\partial^2 G}{\partial x^2} b^2 \delta t$$

Taking Limits

Taking limits $$dG = \frac{\partial G}{\partial x} dx + \frac{\partial G}{\partial t} dt + ½ \frac{\partial^2 G}{\partial x^2} b^2 dt$$

Substituting $$dx = a\ dt + b\ dz$$

We obtain $$dG = \left(\frac{\partial G}{\partial x} a + \frac{\partial G}{\partial t} + ½ \frac{\partial^2 G}{\partial x^2} b^2 \right) dt + \frac{\partial G}{\partial x} b\ dz$$

This is Ito's Lemma

Application of Ito's Lemma to a Stock Price Process

The stock price process is

$$dS = \mu S\, dt + \sigma S\, dz$$

For a function G of S and t

$$dG = \left(\frac{\partial G}{\partial S} \mu S + \frac{\partial G}{\partial t} + ½ \frac{\partial^2 G}{\partial S^2} \sigma^2 S^2 \right) dt + \frac{\partial G}{\partial S} \sigma S\, dz$$

Examples

1. The forward price of a stock for a contract maturing at time T

$$G = S\, e^{r(T-t)}$$

$$dG = (\mu - r)G\, dt + \sigma G\, dz$$

2. $G = \ln S$

$$dG = \left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma\, dz$$

Black-Scholes Model

As $\Delta S/S = \mu^*\Delta t + \sigma^*\Delta z$, hence we have:

$$d\log_e S = (\mu - \sigma^2/2)^*dt + \sigma^*dz$$

where we have assumed the stock prices are <u>lognormally distributed</u>, thus:

$$\log_e S_T \sim N[\log_e S_0 + (\mu - \sigma^2/2)^*T \, , \, \sigma^2{}^*T]$$

Assumptions for Black-Scholes Merton model

1) **The stock pays no dividends during the option's life**: Most companies pay dividends to their share holders. A common way of adjusting the model for this situation is to subtract the discounted value of a future dividend from the stock price.
2) **European exercise terms are used**: European exercise terms dictate that the option can only be exercised on the expiration date. American exercise term allow the option to be exercised at any time during the life of the option, making American options more valuable due to their greater flexibility. This limitation is not a major concern because very few calls are ever exercised before the last few days of their life. This is true because when you exercise a call early, you forfeit the remaining time value on the call.

Assumptions for Black-Scholes Merton model

3) **Markets are efficient**: This assumption suggests that people cannot consistently predict the direction of the market or an individual stock. The market operates continuously with share prices following a continuous Itô process. To understand what a continuous Itô process is, you must first know that a Markov process is "one where the observation in time period t depends only on the preceding observation." An Itô process is simply a Markov process in continuous time. If you were to draw a continuous process you would do so without picking the pen up from the piece of paper.
4) **No commissions are charged**: Usually market participants do have to pay a commission to buy or sell options. Even floor traders pay some kind of fee, but it is usually very small. The fees that individual investor's pay is more substantial and can often distort the output of the model.

Assumptions for Black-Scholes Merton model

5) **Interest rates remain constant and known**: The Black and Scholes model uses the risk-free rate to represent this constant and known rate. In reality there is no such thing as the risk-free rate, but the discount rate on Government Treasury Bills with 30 days left until maturity is usually used to represent it. During periods of rapidly changing interest rates, these 30 day rates are often subject to change, thereby violating one of the assumptions of the model.
6) **Returns are lognormally distributed**: This assumption suggests, returns on the underlying stock are normally distributed, which is reasonable for most assets that offer options.

Black-Scholes Merton differential equation

$$\frac{\partial f}{\partial t} + r_f S \frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial^2 S^2} = r_f f$$

Where:

f : Value of the Forward contract

S : Stock price

r_f : Risk free interest rate

t : time

σ : Volatility of the underlying Asset

The model describes the evolution of the price of an option over time, as a function of time and the value of the underlying. The assumption of normality is very important as it leads to under-pricing in non-gaussian situations. Must be used with caution.

Black-Scholes Merton differential equation

Method for calculation the put and call price can be checked using a good financial calculator available now a days or any good site on the net. An example is

http://www.blobek.com/black-scholes.html